Name: ____________________

Student Number: ____________________

Section: ____________________

Date: ____________________

EE204 - Applied Linear Algebra
Exam 2, Form: $\boxed{\text{A}}$

The exam contains only short answer questions.

Total number of points in this exam = 100

Section 1. Short Answers-A-10 questions, 100 points, 10 points each

1. For which right side $\vec{b}$ does the system have a solution?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 10 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2. Find the projection matrix P that takes any vector $\vec{b}$ into $C(A)$ the column space of the matrix A. Take $A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. If $\vec{b} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ find its projection onto $C(A)$ and the distance from the vector $\vec{b}$ to the subspace $C(A)$.

3. Find the reduced row echelon form R for the 3 by 4 matrix A, having $a_{ij} = 2i + j - 1$

4. Suppose S is spanned by the vectors (1,1,1,1) and (-2,0,0,-2). Find two vectors that span $S^{\perp}$.

5. Construct a matrix whose column space contains (1,2,2), (2,1,1) and (3,3,4) and whose nullspace contains the vector (-23/3,-17/3,5,1).

6. Given the vectors $\vec{v_1} = (1, 2, 2, -3), \vec{v_2} = (2, 1, 1, 1), \vec{v_3} = (3, 3, 4, 4)$ and $\vec{v_4} = (2, 10, 13, -2)$. Test them for being linearly independent or not? Show all your work.

7. The LU decomposition of the matrix A is given by:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

   Without finding A, write down a basis for the column space of A and a basis for the row space of A. Explain your answers.

8. Find the complete solution ($\vec{x_c} = \vec{x_p}$+any multiple of the special solutions $\vec{s}$) to the system

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

   given that

9. Find values for a, b so that the following matrix has orthogonal columns:

$$A = \begin{bmatrix} -1 & +1 & a \\ 1 & 3 & 1 \\ -2 & 1 & b \end{bmatrix}$$

10. Find the LU factorization of the <u>*rectangular*</u> matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Answer Key for Exam $\boxed{\text{A}}$

Section 1. Short Answers-A-10 questions, 100 points, 10 points each

1. For which right side $\vec{b}$ does the system have a solution?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 10 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer: $aug = [Ab] = \begin{bmatrix} 1 & 3 & b_1 \\ 2 & 10 & b_2 \\ -3 & -9 & b_3 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 4 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_1 \end{bmatrix}$ and the condition $b_3 + 3b_1 = 0$.

$RHS = \begin{bmatrix} b_1 \\ b_2 \\ -3b_1 \end{bmatrix}$

2. Find the projection matrix P that takes any vector $\vec{b}$ into $C(A)$ the column space of the matrix A. Take $A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. If $\vec{b} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ find its projection onto $C(A)$ and the distance from the vector $\vec{b}$ to the subspace $C(A)$.

Answer: $P = A * inv(A' * A) * A', \vec{p} = P * \vec{b}, \vec{e} = \vec{b} - \vec{p}.$

$A' * A = 3, inv(A' * A) = \frac{1}{3}$. $P = \frac{1}{3} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$\vec{p} = PA = [-2; 2; -2]$
$\vec{e} = \vec{b} - \vec{p} = [0; 3; 3]$

3. Find the reduced row echelon form R for the 3 by 4 matrix A, having $a_{ij} = 2i + j - 1$

Answer:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -2 & -4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

,

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Suppose S is spanned by the vectors (1,1,1,1) and (-2,0,0,-2). Find two vectors that span $S^\perp$.

Answer: $S^\perp$ is in the nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 0 & -2 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.
The vectors $\vec{s_1} = (-1, 0, 0, 1)$ and $\vec{s_2} = (0, -1, 1, 0)$ span $S^\perp$.

5. Construct a matrix whose column space contains (1,2,2), (2,1,1) and (3,3,4) and whose nullspace contains the vector (-23/3,-17/3,5,1).

Answer: Let this matrix be $A = \begin{bmatrix} 1 & 2 & 3 & a \\ 2 & 1 & 3 & b \\ 2 & 1 & 4 & c \end{bmatrix}$.

$A\vec{x} = \vec{0}$, $\vec{x} = \begin{bmatrix} -23/3 \\ -17/3 \\ 5 \\ 1 \end{bmatrix}$.

Form the equations:
$-23/3 - 34/3 + 15 + x = 0, \Rightarrow a = 4$
$-46/3 - 17/3 + 15 + b = 0, \Rightarrow b = 6$
$-46/3 - 17/3 + 20 + c = 0, \Rightarrow c = 1$
Then $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 6 \\ 2 & 1 & 4 & 1 \end{bmatrix}$

6. Given the vectors $\vec{v_1} = (1, 2, 2, -3), \vec{v_2} = (2, 1, 1, 1), \vec{v_3} = (3, 3, 4, 4)$ and $\vec{v_4} = (2, 10, 13, -2)$. Test them for being linearly independent or not? Show all your work.

Answer: Form the matrix A from the four vectors as its columns.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 3 & 10 \\ 2 & 1 & 4 & 13 \\ -3 & 1 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 7 & 13 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 6 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find its rref $R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
only 3 pivots, hence they are not linearly independent.

7. The LU decomposition of the matrix A is given by:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Without finding A, write down a basis for the column space of A and a basis for the row space of A. Explain your answers.

Answer: Row space basis (-2,1,5), (0,3,2), (0,0,6). Column space basis (1,-2,3), (0,1,2), (0,0,1).
Linear combinations of the columns of L produce A, hence both A and L have the same column space.
$C(A) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$
basis are $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$
Linear combinations of the rows of U produce A, hence both A and U have the same row space.
$C(A^T) = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 5 & 2 & 6 \end{bmatrix}$
basis are $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$
$A = [-2, 1, 5; 4, 1, -8; -6, 9, 25]$

8. Find the complete solution ($\vec{x_c} = \vec{x_p}$+any multiple of the special solutions $\vec{s}$) to the system

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

given that

Answer: Form the augmented matrix and use row elimination to get $aug = [A : \vec{b}] \Rightarrow [R : \vec{d}]$.

$$aug = \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \Rightarrow [R : \vec{d}] = \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The free column is # 3, then $\vec{x}_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$ (setting element # 3 to 0),

$\vec{x}_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (setting element # 3 to 1) and $\vec{x}_{complete} = \alpha\vec{x}_n + x_p$.

9. Find values for a, b so that the following matrix has orthogonal columns:

$$A = \begin{bmatrix} -1 & +1 & a \\ 1 & 3 & 1 \\ -2 & 1 & b \end{bmatrix}$$

Answer: Take dot products of the first two columns with the third column to get: $-a+1-2b=0; a+3+b=0$. Solving to get $a=-7, b=4$.

10. Find the LU factorization of the <u>*rectangular*</u> matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \Rightarrow_{e_{21}=-2, e_{31}=-3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \Rightarrow_{e_{32}=-2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Name: ____________________

Student Number: ____________________

Section: ____________________

Date: ____________________

EE204 - Applied Linear Algebra
Exam 2, Form: $\boxed{\text{B}}$

$\boxed{\text{The exam contains only short answer questions.}}$

$\boxed{\text{Total number of points in this exam} = 100}$

Section 1. Short Answers-B-10 questions, 100 points, 10 points each

1. Find the complete solution ($\vec{x_c} = \vec{x_p}$+any multiple of the special solutions $\vec{s}$) to the system
$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$
given that

2. Find the LU factorization of the <u>*rectangular*</u> matrix A:
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

3. Find the projection matrix P that takes any vector $\vec{b}$ into $C(A)$ the column space of the matrix A. Take $A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. If $\vec{b} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ find its projection onto $C(A)$ and the distance from the vector $\vec{b}$ to the subspace $C(A)$.

4. Suppose S is spanned by the vectors (1,1,1,1) and (-2,0,0,-2). Find two vectors that span $S^{\perp}$.

5. For which right side $\vec{b}$ does the system have a solution?
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 10 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

6. Construct a matrix whose column space contains (1,2,2), (2,1,1) and (3,3,4) and whose nullspace contains the vector (-23/3,-17/3,5,1).

7. Find the reduced row echelon form R for the 3 by 4 matrix A, having $a_{ij} = 2i + j - 1$

8. Find values for a, b so that the following matrix has orthogonal columns:
$$A = \begin{bmatrix} -1 & +1 & a \\ 1 & 3 & 1 \\ -2 & 1 & b \end{bmatrix}$$

9. Given the vectors $\vec{v_1} = (1, 2, 2, -3), \vec{v_2} = (2, 1, 1, 1), \vec{v_3} = (3, 3, 4, 4)$ and $\vec{v_4} = (2, 10, 13, -2)$. Test them for being linearly independent or not? Show all your work.

10. The LU decomposition of the matrix A is given by:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$
Without finding A, write down a basis for the column space of A and a basis for the row space of A. Explain your answers.

Answer Key for Exam $\boxed{\text{B}}$

Section 1. Short Answers-B-10 questions, 100 points, 10 points each

1. Find the complete solution ($\vec{x_c} = \vec{x_p}$+any multiple of the special solutions $\vec{s}$) to the system

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

given that

Answer: Form the augmented matrix and use row elimination to get $aug = [A : \vec{b}] \Rightarrow [R : \vec{d}]$.

$$aug = \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \Rightarrow [R : \vec{d}] = \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The free column is # 3, then $\vec{x}_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$ (setting element # 3 to 0),

$\vec{x}_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (setting element # 3 to 1) and $\vec{x}_{complete} = \alpha \vec{x}_n + x_p$.

2. Find the LU factorization of the <u>rectangular</u> matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \Rightarrow_{e_{21}=-2, e_{31}=-3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \Rightarrow_{e_{32}=-2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix},$$
$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

3. Find the projection matrix P that takes any vector $\vec{b}$ into $C(A)$ the column space of the matrix A. Take $A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. If $\vec{b} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ find its projection onto $C(A)$ and the distance from the vector $\vec{b}$ to the subspace $C(A)$.

Answer: $P = A * inv(A' * A) * A', \vec{p} = P * \vec{b}, \vec{e} = \vec{b} - \vec{p}$.

$A' * A = 3, inv(A' * A) = \frac{1}{3}$. $P = \frac{1}{3} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$\vec{p} = PA = [-2; 2; -2]$
$\vec{e} = \vec{b} - \vec{p} = [0; 3; 3]$

4. Suppose S is spanned by the vectors (1,1,1,1) and (-2,0,0,-2). Find two vectors that span $S^{\perp}$.

   Answer: $S^{\perp}$ is in the nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 0 & -2 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.
   The vectors $\vec{s_1} = (-1, 0, 0, 1)$ and $\vec{s_2} = (0, -1, 1, 0)$ span $S^{\perp}$.

5. For which right side $\vec{b}$ does the system have a solution?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 10 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   Answer: $aug = [Ab] = \begin{bmatrix} 1 & 3 & b_1 \\ 2 & 10 & b_2 \\ -3 & -9 & b_3 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 4 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_1 \end{bmatrix}$ and the condition $b_3 + 3b_1 = 0$.
   $RHS = \begin{bmatrix} b_1 \\ b_2 \\ -3b_1 \end{bmatrix}$

6. Construct a matrix whose column space contains (1,2,2), (2,1,1) and (3,3,4) and whose nullspace contains the vector (-23/3,-17/3,5,1).

   Answer: Let this matrix be $A = \begin{bmatrix} 1 & 2 & 3 & a \\ 2 & 1 & 3 & b \\ 2 & 1 & 4 & c \end{bmatrix}$.
   $A\vec{x} = \vec{0}$, $\vec{x} = \begin{bmatrix} -23/3 \\ -17/3 \\ 5 \\ 1 \end{bmatrix}$.
   Form the equations:
   $-23/3 - 34/3 + 15 + x = 0, \Rightarrow a = 4$
   $-46/3 - 17/3 + 15 + b = 0, \Rightarrow b = 6$
   $-46/3 - 17/3 + 20 + c = 0, \Rightarrow c = 1$
   Then $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 6 \\ 2 & 1 & 4 & 1 \end{bmatrix}$

7. Find the reduced row echelon form R for the 3 by 4 matrix A, having $a_{ij} = 2i + j - 1$

   Answer:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -2 & -4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   ,

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Find values for a, b so that the following matrix has orthogonal columns:

$$A = \begin{bmatrix} -1 & +1 & a \\ 1 & 3 & 1 \\ -2 & 1 & b \end{bmatrix}$$

   Answer: Take dot products of the first two columns with the third column to get: $-a+1-2b = 0; a+3+b = 0$.
   Solving to get $a = -7, b = 4$.

9. Given the vectors $\vec{v_1} = (1, 2, 2, -3), \vec{v_2} = (2, 1, 1, 1), \vec{v_3} = (3, 3, 4, 4)$ and $\vec{v_4} = (2, 10, 13, -2)$. Test them for being linearly independent or not? Show all your work.

Answer: Form the matrix A from the four vectors as its columns.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 3 & 10 \\ 2 & 1 & 4 & 13 \\ -3 & 1 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 7 & 13 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 6 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find its rref $R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

only 3 pivots, hence they are not linearly independent.

10. The LU decomposition of the matrix A is given by:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Without finding A, write down a basis for the column space of A and a basis for the row space of A. Explain your answers.

Answer: Row space basis (-2,1,5), (0,3,2), (0,0,6). Column space basis (1,-2,3), (0,1,2), (0,0,1).

Linear combinations of the columns of L produce A, hence both A and L have the same column space.

$C(A) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

basis are $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$

Linear combinations of the rows of U produce A, hence both A and U have the same row space.

$C(A^T) = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 5 & 2 & 6 \end{bmatrix}$

basis are $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$

$A = [-2, 1, 5; 4, 1, -8; -6, 9, 25]$